Tighter Upper Bound for the Number of Kobon Triangles

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***Abstract*—What is the maximal number of nonoverlapping triangles realizable by n straight lines in a plane? This problem stated by Fujimura Kobon is an unsolved problem in combinatorial geometry. Saburo Tamura proved that $\lfloor n(n-2)/3 \rfloor$ provides an upper bound on the maximal number. In this paper we present the concept of perfect configuration of lines which are then used to proof that the bound known of Tamura can not be reached for all n with $n \equiv 0 \mod 6$ and $n \equiv 2 \mod 6$.**

In other words a new tighter bound is introduced here which is equal to: $n(n-2)/3$ when $n \mod 6 \in \{3,5\}$, $(n+1)(n-3)/3$ when $n \mod 6 \in \{0,2\}$, $(n-1)^2/3$ when $n \mod 6 \in \{1,4\}$.

***Index Terms*—Kobon triangles, upper bound, combinatorial geometry, math puzzle**

I. Introduction

A *Kobon triangle* is a triangle that is realized by 3 straight lines segments and which does not overlap with other triangles. What is the largest number $K(n)$ of such triangles constructed by drawing n lines in the plane (we hereafter call an arrangement of n straight lines *configuration*)? Kobon Fujimura, a Japanese puzzle expert and math teacher, posed in 1978 this question in his book "The Tokyo Puzzle" [2], [3]. For up to six lines it's easy to find $K(n)$ and the corresponding optimal configurations. However, despite the problem is easy to state, an analytic expression for $K(n)$ is still unknown and believed to be hard to find [4].

Table I shows the number of Kobon triangles realized by the best known configurations of up to n lines. The numbers are listed the *On-Line Encyclopedia of Integer Sequences* as A006066 [1]. Of course every configuration that reaches the upper bound is optimal. In the following we consider only the non degenerated cases of $n \geq 3$.

Although a analytic expression for the number of triangles is unknown, Saburo Tamura has proved that

$$K(n) \leq \left\lfloor \frac{n(n-2)}{3} \right\rfloor \tag{1}$$

is an upper bound on $K(n)$. The proof of Saburo Tamura directly follows from the proof of Lemma 1. The resulting series is registered as A032765 [1]. The first terms can be seen in Table I as well as the configuration that reach the bound (bold). The largest configuration was recently found by one of the author and consists of 17 lines (see Figure 1).

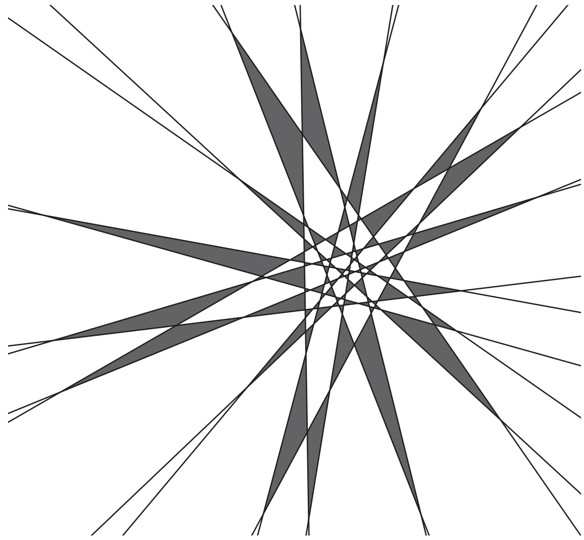

Figure 1. This configuration of 17 lines generates 85 nonoverlapping triangles. According to (1) the solution is optimal.

Most of the known configurations come very close to the upper bound, however, it's noticeable that no configurations with $n \equiv 4 \mod 6$ apart from $n = 4$ is known that reaches the upper bound.

To find a tighter proof, we build up on the following proposition:

Proposition 1. *Let's call the intersection of two or more lines* point *and a part of a line that is bounded by two points* segment. *Given a configuration of n lines, the maximal number of points is $n(n-1)/2$ and the maximal number of segments is equal to $n(n-2)$.*

Proof: If all straight lines are pairwise intersecting and three different lines don't intersect a same common point then every line intersects the remaining $n-1$ lines once which divides the line into $n-2$ segments. The total number of line segments sums up to $n(n-2)$ and the number of intersection points to $n(n-1)/2$. □

II. Main Proof

Definition 1. *A* perfect configuration *is an arrangement of n pairwise intersecting lines, where each segment is the side*

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$(n \mod 6)$	3	4	5	0	1	2	3	4	5	0	1	2	3	4	5	0	1	2
$K(n)$	**1**	**2**	**5**	7*	**11**	15*	**21**	25	32	38	**47**	53	**65**	72	**85**	93	104	115
Bound of Tamura	1	2	5	8	11	16	21	26	33	40	47	56	65	74	85	96	107	120
Authors' bound	1	2	5	7	11	15	21	26	33	39	47	55	65	74	85	95	107	119

Table I
THE NUMBER OF NONOVERLAPPING TRIANGLES $K(n)$ THAT THE BEST KNOWN CONFIGURATIONS OF n STRAIGHT LINES REALIZES. THE BOLD NUMBERS REACH THE BOUND OF TAMURA, THE STARS INDICATE THE CONFIGURATION WHICH REACH THE TIGHTER BOUND (2) AND THEREFORE ARE MAXIMAL AS WELL.

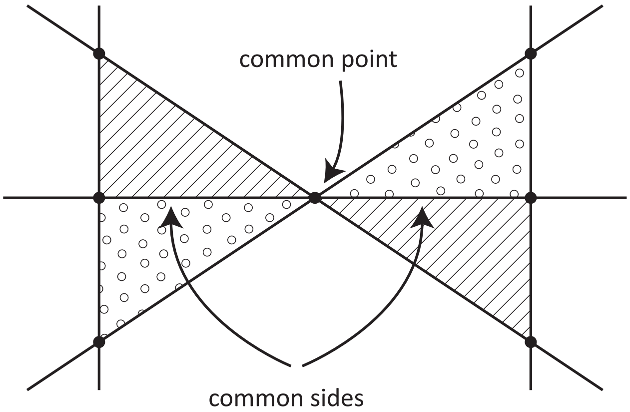

Figure 2. Two pairs of triangles share one common side each and a common point which is the intersection of (at least) three lines.

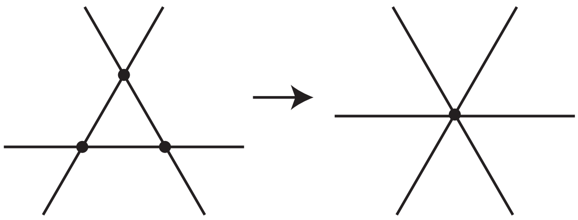

Figure 3. The number of line segments and points decreases by 3 and 2 respectively if a line intersects an existing point.

of exactly one nonoverlapping triangle and $K(n)$ meets the upper bound (1).

Lemma 1. *If* $(n \mod 3) \in \{0, 2\}$*, then all configurations that meet the upper bound* (1) *are perfect configurations. In these cases* $n(n-2) \equiv 0 \mod 3$ *hence* $K(n) = n(n-2)/3$.

Proof: We have to show that no configuration with common side triangles and $(n \mod 3) \in \{0, 2\}$ exists with $K(n) = \frac{n(n-2)}{3}$. This is equal to show that the number of line segments needed is larger than $3K(n)$. We proof the claim that a triangle needs three line segments if they don't side another triangle and more in all other cases. By looking at Figure 2 we see:

1) A line segment can be the side of one or two triangles.
2) If a line segment is the side of two triangles then the corresponding line intersects at one of the two endpoints an existing point which belongs to both triangles.
3) From 2. follows that every intersection point with more than two corresponding lines is part of at most two pairs of triangles that share a common side.

By looking at Figure 3 we see that if a line intersects an existing point then the number of points decreases by 2 and the number of segments by 3. Hence we "save" at most two segments (the common ones) but we loose three due to the intersection of more than two lines in one point. Hence the number of line segments needed to build triangles increases if one side belongs to two triangles. Clearly, if a side does not belong to any triangle the number increases as well. Therefore the number of line segments needed is minimized if each line segment belongs to exactly one triangle and cannot be reached in all other cases. □

Definition 2. *The first and the last point on a line are called* extremal point*. The* degree *of a point is the number of segments which are connected to such a point.*

Lemma 2. *In a perfect configuration all extremal points have degree 2.*

Proof: In a perfect configuration not more than two lines intersect in the same point because otherwise the number of line segments would decrease according to the Proof to Lemma 1 and the upper bound could not be reached. There are only three possible degrees for any point P in a figure: 2, 3 or 4 because:

- Degrees smaller than 2 are not possible as on both lines of P there are $n(n-1) \geq 2$ points and therefore at least two line segments attached to P.
- Degrees bigger than 4 are not possible as on each of the two lines there can be at most one segment on the left of P and one segment on the right.

If a point P is of degree 4 then both corresponding lines have two points on the left and the right of P and P is an extremal point of neither of them. Degrees of 3 are as well not possible because then the configuration cannot be perfect as can be seen from Figure 4: If the configuration is perfect then BP and PC must be siding a triangle as per Definition 1. Hence ABP and ACP must be triangles. Then AP is siding two triangles which contradicts Definition 1. □

Lemma 3. *A perfect configuration exists only for odd n.*

Proof: Lets consider one line L of the perfect configuration (see Figure 5). As per Proposition 1 the line is divided in $n-2$ line segments. Each of this segments belongs to exactly one triangle as per Definition 1. Starting with the

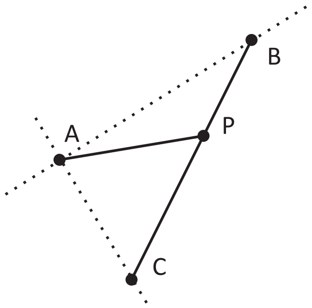

Figure 4. No extremal point P in a perfect configuration is of degree 3. If the figure is maximal then BP and PC must be siding a triangle as per Definition 1. Hence ABP and ACP must be triangles. Then AP is siding two triangles which is contradicting Definition 1.

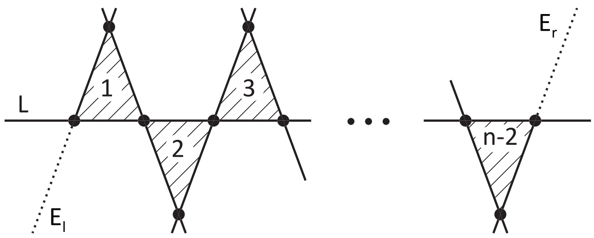

Figure 5. The line L consists of $n-2$ segments which side triangles that lie alternately above and below L. Since E_l and E_r intersect in a point on one of the two dotted rays for $n \equiv 0 \mod 2$, one of the extremal points of L has degree 3 which contradicts Lemma 2

leftmost segment we assume without loss of generality that the corresponding triangle lies above of L (triangle 1). By looking at Figure 4 we see that the second triangle needs to be below L - otherwise the first and the second triangle would have a common side - and so on. If n is odd then the last triangle lies below L.

Now lets have a look at the two lines E_l and E_r that intersect the two extremal points of L. Since in a perfect configuration no lines are parallel, E_1 and E_2 intersect either above or below L. This entails a point on the dotted ray of either E_l or E_r. But this in turn means that the corresponding extremal point on L is of degree 3 which contradicts Lemma 2. □

Theorem 1. *The maximal number of Kobon triangles $K(n)$ for a given number of n straight lines in a plane is upper bounded by*

$$K(n) \leq \left\lfloor \frac{n(n-2)}{3} \right\rfloor - \mathbf{I}_{\{n|(n \mod 6) \in \{0,2\}\}}(n) \quad (2)$$

where $\mathbf{I}_A(x)$ denotes the indicator function. In other words the upper bound known by Saburo Tamura cannot be reached for all n with $n \equiv 0 \mod 6$ and $n \equiv 2 \mod 6$.

Proof: According to Lemma 1 the upper bound can only be reached by perfect configurations if $n \equiv 0 \mod 3$ or $n \equiv 2 \mod 3$. But these perfect configurations are only possible for odd n according to Lemma 3. Therefore for $n \mod 3 \in \{0,2\}$ and $n \equiv 0 \mod 2$ the upper bound cannot be reached. These two conditions can be summarized as $n \mod 6 \in \{0,2\}$.

In other words the upper bound is

$$B(n) = \begin{cases} n(n-2)/3 & n \mod 6 \in \{3,5\} \\ n(n-2)/3 - 1 & n \mod 6 \in \{0,2\} \\ (n-1)^2/3 & n \mod 6 \in \{1,4\} \end{cases} \quad (3)$$

□

The last row of Table I shows the new bound. The stars in the third line indicate the configurations that are optimal in addition to the already known optimal solutions (bold).

References

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